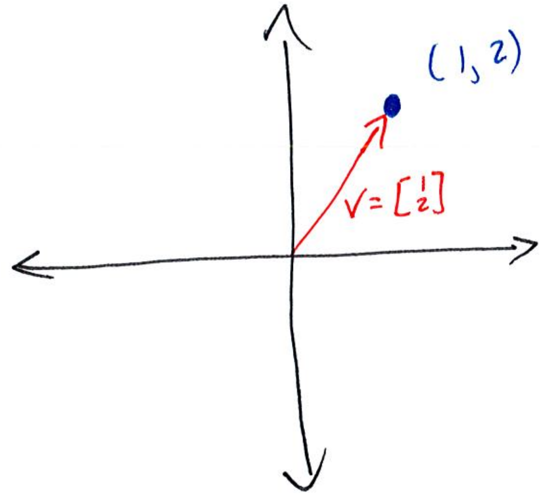
§1.3 Vector Equations

Quick review of vectors in $\mathbb{R}^2$, the real plane. As an example, take $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ which is a <u>(column) vector</u>. Geometrically,



Defining properties are direction and length.

Notice that $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a 2×1 matrix, hence $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \end{bmatrix}$ as these are matrices of different sizes. However, we can associate vectors in $\mathbb{R}^2$ to points so we can say

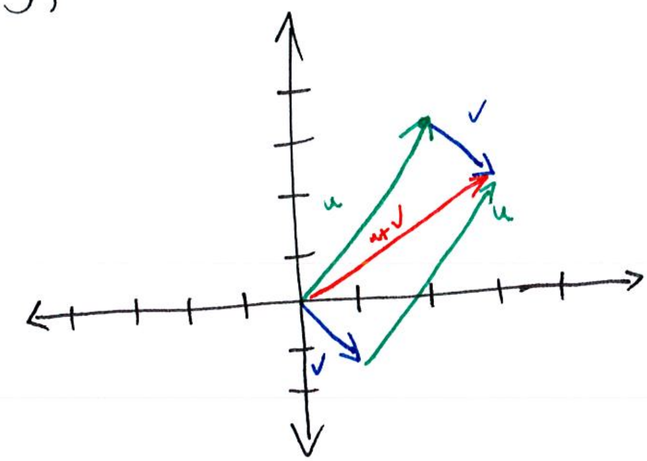
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{``}=\text{''} (1, 2)$$

Vectors in $\mathbb{R}^2$ have nice algebraic and geometric properties. For example, let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then $u + v = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 3-1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Geometrically,

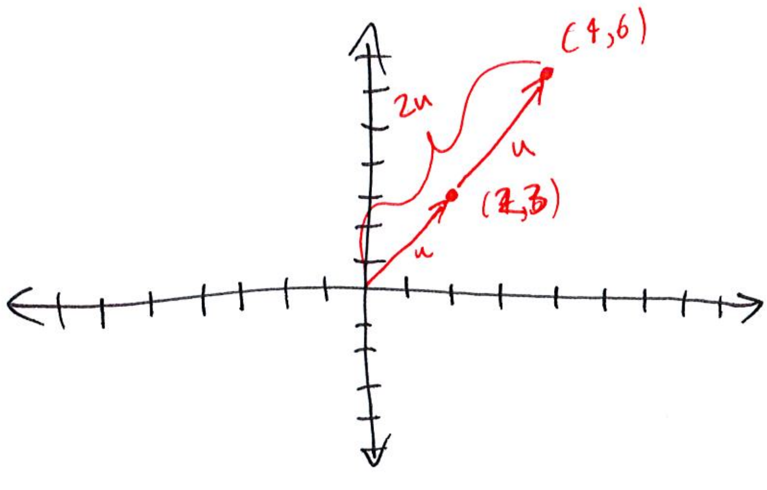


We also have the notion of scalar multiplication. If $v = \begin{bmatrix} a \\ b \end{bmatrix}$ and c is a real number, then

$c \cdot v = c \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$. For example

with $u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ as before

$2u = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$. Geometrically



Notice $2u$ is the same as $u + u$. (which algebraically it should be)

Now generalize from $\mathbb{R}^2$ to $\mathbb{R}^n$ whose elements are $n \times 1$ column vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$ where $u_1, \ldots, u_n$ are real numbers.

The same addition/scalar multiplication properties hold (but they're much harder to see geometrically when $n > 3$!)

As a matter of notation we denote the zero vector $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ in $\mathbb{R}^n$ by 0.

It will be clear from context when using 0 as the zero vector and 0 as the real number.

Defn Given vectors $v_1, v_2, \ldots, v_m$ in $\mathbb{R}^n$ any vector u given by

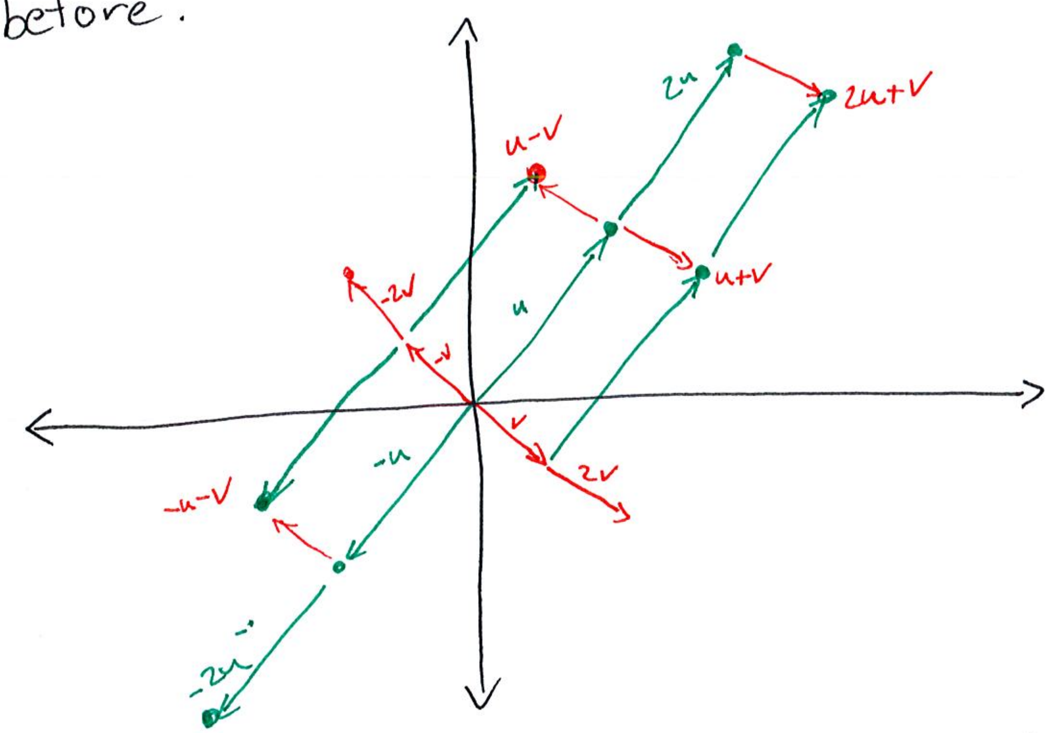
$$u = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

for real numbers $c_1, \ldots, c_m$ is called a Linear Combination of $v_1, \ldots, v_m$.

Back in $\mathbb{R}^2$, sets of linear combinations have nice geometric realizations. Let

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

as before.



Creating a net or lattice of the plane. Notice we only took integer multiples of u, v above but could use all real numbers.

<u>Defn</u> If $v_1, \ldots, v_m$ are vectors in $\mathbb{R}^n$, the <u>span</u> of $v_1, \ldots, v_m$ is the set of all linear combinations of $v_1, \ldots, v_m$. We denote it by $\text{Span}\{v_1, \ldots, v_m\}$

Topic of interest: Given vectors $v_1, \ldots, v_m$ and b of $\mathbb{R}^n$, is there a way to tell if b is in $\operatorname{Span}\{v_1, \ldots, v_m\}$?

This is equivalent to asking if be can be written as

$$b \stackrel{?}{=} c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$$

In other words, is there a solution to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_m v_m = b$$

or equivalently, is there a solution to the linear system whose augmented matrix is

$$\left[\begin{array}{c|c|c|c|c} v_1 & v_2 & \cdots & v_m & b \end{array}\right]$$

which we know how to do!

Example

Is $b = \begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix}$ in $\text{span}\left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}} \right\}$?

In matrix form $[v_1 \ v_2 \mid b] = \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 4 & 3 & 9 \end{array}\right]$

$$\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 3 & 1 & 5 \\ 4 & 3 & 9 \end{array}\right] \xrightarrow[-4R_1 + R_3 \to R_3]{-3R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -7 \\ 0 & -5 & -7 \end{array}\right]$$

$$\xrightarrow{-R_2 + R_3 \to R_3} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & -5 & -7 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 7/5 \\ 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 6/5 \\ 0 & 1 & 7/5 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow \begin{cases} x_1 = 6/5 \\ x_2 = 7/5 \end{cases}$$

yes! This ~~tells~~ tells us that

$$\begin{bmatrix} 4 \\ 5 \\ 9 \end{bmatrix} = \frac{6}{5}\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \frac{7}{5}\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Thus b is in $\text{span}\{v_1, v_2\}$